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Definitions:

- **Linear equation**: an equation that can be written in the form $a_1x_1 + \ldots + a_nx_n = b$
  - **System of linear equations**: collection of one or more linear equations involving the same variables
- **Solution**: a list of numbers that makes each equation in a system of linear equations a true statement when its values are substituted in for $x_1 \ldots x_n$
  - **Solution set:** set of all possible solutions of a linear system
    - **Equivalent**: when two linear systems have same solution set
- **Consistent**: when a system of linear equations has one solution or infinitely many solutions
  - **Inconsistent**: when a system of linear equations has no solution
- **Linear combination of $\mathbf{v_1} \ldots \mathbf{v_p}$**: the vector ($\mathbf{y}$) defined by $\mathbf{y} = c_1\mathbf{v_1} + \ldots + c_n\mathbf{v_n}$
  - **Weights:** $c_1 \ldots c_p$
  - A vector equation $x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n} = \mathbf{b}$, which has the same solution set as the augmented matrix $[\ \mathbf{a_1} \ldots \mathbf{a_n}\ \ \mathbf{b}\ ]$.
    - In particular, **b** can be generated by a linear combination of $\mathbf{a_1} \ldots \mathbf{a_n}$ if and only if there exists a solution to the linear system corresponding to the matrix or the left hand of the above vector equation
- **Span**: all valid solutions to a vector equation $\mathbf{b} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_n\mathbf{v_n}$
  - Asking whether a vector **b** is in **Span $\{\mathbf{v_1} \ldots \mathbf{v_p}\}$** is equivalent to asking whether the equation $x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_n\mathbf{v_n} = \mathbf{b}$ – or the augmented matrix $[\ \mathbf{v_1}\ \ldots\ \mathbf{v_p}\ \ \mathbf{b}\ ]$ – has a solution.
  - Set of all linear combinations
  - ***Span*** **$\{\mathbf{v_1}, \mathbf{v_2} \ldots \mathbf{v_p}\}$ contains every scalar multiple of every vector in that span as a solution because** $c\mathbf{v_1} = c\mathbf{v_1} + 0\mathbf{v_2} + \ldots + 0\mathbf{v_p}$
- $A\mathbf{x} = \mathbf{b}$: If A is an *m* x *n* matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and if **x** is in R^n, then the **product of** A **and x**, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in x as weights.
  - $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries (rows) in **x**
  - $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation $= x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_n\mathbf{v_n} = \mathbf{b}$ and the augmented matrix $[\ \mathbf{a_1}\ \ \mathbf{a_2} \ldots \mathbf{a_n}\ \ \mathbf{b}\ ]$
  - The equation $A\mathbf{x} = \mathbf{b}$ has a solution if an only if **b** is a linear combination of the columns of A
    - Asking if **b** is in span **$\{\mathbf{a_1}, \ldots, \mathbf{a_n}\}$ is the same as asking is** $A\mathbf{x} = \mathbf{b}$ consistent
- **Homogeneous**: when a system of linear equations can be written as $A\mathbf{x} = \mathbf{0}$ – *when the right hand side is all zeros*.
  - Always has one **trivial solution**, where $\mathbf{x} = \mathbf{0}$
    - **Non-trivial solution**: solution where *at least one* entry in **x** is nonzero
  - The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a **nontrivial solution** if and only if the equation has at least one free variable

- To summarize the two cases: one unique solution that has to be the trivial solution, or infinitely many solutions

- **Nonhomogeneous:** Ax = **b**, **b** $\neq 0$
- **Diagonal entries:** The entries in an *m* x *n* matrix **A** = [$\mathbf{a_{ij}}$] following the pattern a_{11}, a_{22}, a_{33} …
  - These entries form the **main diagonal** of **A**
- **Diagonal matrix:** a square *n* x *n* matrix whose non-diagonal entries are zero
  - An example is the *n* x *n* identity matrix, I_n
- **Identity matrix:** an *n* x *n* matrix of all zeros, except for the main diagonal, containing all ones
  - Any *n* x *n* invertible matrix A times its inverse A^{-1} equals an *n* x *n* identity matrix I_n
- **Elementary matrix:** A matrix obtained by performing a single elementary row operation on an identity matrix
- **Zero matrix**: an *m* x *n* matrix that contains all zeros
- **Addition/subtraction**: Add or subtract corresponding entries in two or more matrices. To add and subtract, matrices must be the same size.
- **Scalar multiplication**: multiply every entry in a matrix by the given scalar
- **Matrix multiplication**: if A is an *m* x *n* matrix and B is an *n* x *p* matrix, then AB is an *m* x *p* matrix where each entry, C_{ij}, is the dot product of the i^{th} row of A and j^{th} column of B
- **Power:** A matrix times itself a given number of times. *Only works for square matrices!*
- **Invertible:** A is **invertible** when there exists an *n* x *n* matrix C such that CA = I and AC = I, where I = I_n – the identity matrix
  - C is the **inverse** of A, denoted by A^{-1}
    - $A^{-1}A = I$ and $AA^{-1} = I$
  - In general, we only consider square matrices when talking about inverses, but not every square matrix is invertible
  - **Singular**: a matrix that is not invertible
    - If the determinant of the matrix is 0, then that matrix is singular.
    - **Linearly dependent**
  - **Nonsingular**: a matrix that is invertible
    - **Has only the trivial solution for Ax = 0**
    - **The columns of A are linearly independent.**
    - **Ax = b has exactly one solution for each b in Kn. UNIQUE**
    - **Row equivalent to identity matrix**
    - **n pivot positions!!!!**
    - **Columns of A span R^n**
    - **A^T is invertible**
    - **$A^{-1}$is invertible $(A^{-1})^{-1} = A$**
  - Simple formula for the inverse of a 2 x 2 matrix:

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \text{ If } ad - bc \neq 0, \text{ then } A \text{ is invertible and}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{If } ad - bc = 0, \text{ then } A \text{ is not invertible.}$$

- **Important**: If A is an invertible *n* x *n* matrix, then for each **b** in R^n – i.e. each possible solution to the matrix – $\mathbf{x} = A^{-1}\mathbf{b}$
- **Properties of the inverse:**
  - If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
  - If A and B are *n* x *n* invertible matrices, then so is AB, and the inverse of AB is the *product of the inverses of A and B in the reverse order* (recall that AB ≠ BA necessarily and A does not necessarily equal B): $(AB)^{-1} = B^{-1} A^{-1}$
    - *The product of n x n invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order*
    - If AB is invertible, then det(AB) is not equal to zero. Therefore since det(AB)=det(A)*det(B), neither det(A) nor det(B) can be zero, hence both A and B are invertible.
  - If A is an invertible matrix, then so is A^T and the inverse of A^T is the transpose of A^{-1}. That is: $(A^T)^{-1} = (A^{-1})^T$
- **To find A^{-1}:** row reduce the augmented matrix [A I] until we get [I A^{-1}]

- **Determinant:** for n ≥ 2, the determinant of an *n* x *n* matrix A = $[a_{ij}]$ is the sum of *n* terms of the form $\pm a_{1j}$ with plus or minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.
  - **Properties:**
    - det(AB) = det(A) * det(B)
    - det(A^T) = det(A)
    - det(A^{-1}) = 1/det(A)
    - Row Operations:
      - If a multiple of one row of A is added to produce another matrix B (replacement), then det(B) = det(A)
      - If two rows of A are interchanged to produce B, then det(B) = -det(A)
      - If one row of A is multiplied by *k* to produce B, then det(B) = *k* * det(A)
  - **Cramer's Rule**
    - Let A be an invertible *n* x *n* matrix. For any **b** in R^n, the unique solution **x** of A**x** = **b** has entries given by:

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n$$

- **Vector Space:** a nonempty set V of objects, called *vectors*, where addition and multiplication by scalars are defined
  - **Subspace:** a subset H of V where the zero vector of V is in H, H is closed under vector addition (for each **u** and **v** in H, **u** + **v** is in H), and is closed under multiplication by scalars (for each **u** in H, c**u** is in H)
    - <u>If $\mathbf{v_1 ... v_p}$ in vector space V, then span $\{\mathbf{v_1 ... v_p}\}$ is a subspace of V</u>
- **Basis**: an indexed set of vectors B = $\{\mathbf{b_1 ... b_p}\}$ **in** subspace V is a basis for H if B is a linearly independent set and span $\{\mathbf{b_1 ... b_p}\}$ = H
- **Null Space**: set of all solutions to the homogenous equation A**x** = **0**
  - Usually described implicitly, but can be written explicitly by decomposing the equations and putting them into parametric vector form
    - The
- **Column Space**: set of all linear combinations of the columns of A
  - <u>*The pivot columns of A form a **basis** for col A*</u>
- **Transformation (or function or mapping) T:** from R^n to R^m is a rule that assigns to each vector **x** in R^n a vector T(**x**) in R^m
  - **Transformation Lingo:**
    - <u>*Domain of T*</u>: The set R^n
    - <u>*Codomain of T*</u>: The set R^m
    - <u>*Image of x*</u>: The vector T(**x**) in R^m corresponding to an **x** in R^n
      - <u>*Range of T:*</u> set of all images T(**x**)
  - **Linear:** A transformation is linear if T(r**u** + s**v**) = rT(**u**) + sT(**v**) for all **u, v** in the domain of T (e.g. in R^n)
  - **Standard Matrix for Linear Transformation**: A modified identity matrix that transforms any **x**
  - **Important Uniqueness Stuff:**
    - <u>***Onto***</u>: A mapping T: $R^n \rightarrow R^m$ is said to be <u>onto</u> R^m if each **b** in R^m is the image of <u>***at least one***</u> **x** in R^n
      - A transformation T maps R^n onto R^m if and only if the columns of the standard matrix A span R^m
        - When the range of T is all of the codomain R^m (no empty parts of the image)
      - <u>***Implications***</u>:
        - *Range* = R^m (column space of A – aka all linear combinations of A's columns)
          - <u>*Every **b** in R^m is a linear combination of columns of A using the original **x** vector being transformed as a weight*</u>
          - Rank = m
          - Dim(kernel) = $n - m \geq 0$

        - ***<u>One-to-one</u>***: A mapping T: $R^n \rightarrow R^m$ is said to be <u>one-to-one</u> if each **b** in R^m is the image of ***<u>at most one</u>*** **x** in R^n (e.g. there can be gaps)
            - ***<u>Implications</u>***:
                - *Ax = 0 has only trivial solution because only one thing we can map to zero*
                    - *0 in* R^n = 0 in R^m because <u>dimension of kernel is zero</u>
                    - *Columns are linearly independent*
                    - Rank = *n*, because dimension of kernel is zero
                    - $n \leq m$
        - ***<u>One-to-one and onto</u>***: A mapping T: $R^n \rightarrow R^m$ is said to be <u>one-to-one and onto</u> if each **b** in R^m is the image of ***<u>exactly one x</u>*** in R^n
            - ***<u>Implications</u>***:
                - A is a square matrix and A**x** = **b** has a unique solution for every **b** in R^m
                    - A has a unique inverse A^{-1} that is an inverse map that does the reverse
                    - $\det(A) \neq 0$
    - **Kernel**: Given a linear transformation L: $R^n \rightarrow R^m$, the kernel of L is the set of vectors **x** in T: R^n such that L(**x**) = **0** (e.g. null space)
        - Essentially nul A, where A is the standard matrix
        - dim(kernel) = # free variables/# non-pivot columns
    - **Rank**: the dimension of the column space of A
        - # of pivot columns of A, where A is the standard matrix
- **Rank Theorem:**
    - rank + dim(kernel) = n (# of columns)