# Diagonalisation Review Sheet. Math 54, Fall 2012

This is a review of the diagonalisation process for square $n \times n$ matrices $A$.

**Main things to know**

- eigenstuff: eigenvalues, eigenvectors (must be nonzero!), eigenspaces, characteristic polynomial.
- 0 is an eigenvalue of $A$ if and only if $A$ is not invertible.
- if $\{v_1, \dots, v_k\}$ are eigenvectors of $A$ with associated eigenvalues $\lambda_1, \dots, \lambda_k$ such that $\lambda_i \neq \lambda_j$, for $i \neq j$, then $\{v_1, \dots, v_k\}$ is linearly independent. *"distinct eigenspaces of A are linearly independent"*
- if $A$ has $n$ distinct eigenvalues then $A$ is diagonalisable. *"n distinct eigenvalues $\Longrightarrow$ diagonalisable"*
- *Criterion of Diagonalisability (theoretical):* $A$ is diagonalisable $\Leftrightarrow$ there exists a basis of eigenvectors of $A$
- *Criterion of Diagonalisability (practical):* $A$ is diagonalisable $\Leftrightarrow$ for each eigenvalue $\lambda_i$ of $A$ we have

  $$\dim nul(A - \lambda_i) = n_i,$$

  where $n_i$ is the exponent appearing in the characteristic polynomial

  $$(\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$

- *Criterion of diagonalisability (matrix version):* $A$ is diagonalisable $\Leftrightarrow$ there is invertible $P$ such that $P^{-1}AP = D$, where $D$ is diagonal.

  There are some interesting things to note here:

  i) the diagonal entries of $D$ are precisely the eigenvalues $\lambda_i$ of $A$ appearing $n_i$ times and in some order (the order is determined by the columns of $P$),
  ii) the columns of $P$ consists of eigenvectors of $P$,
  iii) if $p_j$ is the $j^{th}$ column of $P$ - therefore an eigenvector of $A$ with associated eigenvalue $\lambda_j$ - then the $j^{th}$ entry on the diagonal of $D$ is $\lambda_j$

**Example** 1. Consider the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of $A$ is

$$\begin{aligned}
\det(A - \lambda I_3) &= \det \begin{bmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \\
&= -\lambda\left(\lambda^2 - 1\right) + (\lambda - 1) + (-1 + \lambda) \\
&= -\lambda(\lambda - 1)(\lambda + 1) + 2(\lambda - 1) \\
&= (\lambda - 1)\left(2 - \lambda(\lambda + 1)\right) = (\lambda - 1)\left(2 - \lambda - \lambda^2\right) = (\lambda - 1)^2(2 + \lambda)
\end{aligned}$$

So that the eigenvalues of $A$ are $\lambda = 1, 1, -2$

In order that $A$ be diagonalisable we must have

$$\dim nul(A - I_3) = 2, \text{ and } \dim nul(A + 2I_3) = 1.$$

Let's check to see:

$\lambda = 1$: we have

$$A - I_3 = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that since there are two free variables we have $\dim nul(A - I_3) = 2$.

$\lambda = -2$: we have

$$A + 2I_3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that since there is one free variable we have $\dim nul(A + 2I_3) = 1$. Hence, we see that $A$ is diagonalisable.

Let's determine $P$ such that $P^{-1}AP = D$, where $D$ is a diagonal matrix. We need to find a basis of eigenvectors of $A$ - using what we have already found we see that

$$nul(A - I_3) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \; nul(A + 2I_3) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Thus, we set

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

to obtain

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

If we set

$$R = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

then we have

$$R^{-1}AR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then, the characteristic polynomial of $A$ is

$$(1 - \lambda)^2(2 - \lambda),$$

so the eigenvalues of $A$ are $\lambda = 1, 1, 2$.

In order for $A$ to be diagonalisable we must have

$$\dim nul(A - I_3) = 2, \quad \text{and} \quad \dim nul(A - 2I_3) = 1.$$

We see that

$$A - I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $\dim nul(A - I_3) = 1 \neq 2$. Hence, $A$ is not diagonalisable.

**Exercises** Determine the eigenstuff and whether the following matrices are diagonalisable:

i) $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$,

ii) $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$,

iii) $A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -1 & 0 \\ -2 & 1 & 2 \end{bmatrix}$,

iv) $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$,

v) $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$,

vi) $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

$i), iii), iv), vi)$ are diagonalisable -why? Why are the remaining matrices not diagonalisable?

**Counterexamples**

- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ - **not invertible, not diagonalisable**
- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ - **not invertible, diagonalisable**
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ - **invertible, diagonalisable**
- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ - **invertible, not diagonalisable**
- $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ are both diagonalisable but $A + B$ is **not diagonalisable**.