Math 54 Final Exam Review

Chapter 1: Linear Equations in Linear Algebra (Sections 1,2,3,4,5,7,8,9)

Section 1: Systems of Linear Equations

- Two systems of equations are **equivalent** if they have the same solution set
- System of equations is **consistent** if it has one or infinitely many solutions; it is **inconsistent** if it has no solutions
- $m \times n$ matrix: m rows, n columns
- **Elementary row operations** (they are reversible)**:**
    - **Replacement:** add to a row a multiple of another row
    - **Interchange:** interchange two rows
    - **Scaling:** multiply all entries in a row by a nonzero constant
- Two matrices are **row equivalent** if there exists a sequence of elementary row operations that transforms one matrix into the other; they have the same solution set
- 2 fundamental questions: existence and uniqueness

Section 2: Row Reduction and Echelon Forms

- **Leading entry:** leftmost nonzero entry in a nonzero row
- **Row echelon form** of a matrix:
    - All nonzero rows are above any rows of zeros
    - Each leading entry of a row is in a column to the right of the leading entry of the row above it
    - All entries in a column below the leading entry are zero
- **Reduced row echelon form** of a matrix:
    - Leading entry in each nonzero row is 1
    - Each leading 1 is the only nonzero entry in its column
- Each matrix can be reduced down to multiple different matrices in echelon form, but is only row equivalent to <u>one</u> matrix in reduced row echelon form.
- A **pivot position** is a location that corresponds to a leading 1 in RREF of the matrix. A **pivot column** is the column of A that contains a pivot position.
- A linear system is consistent iff its augmented matrix contains no rows [0 0 … 0 | b] with a nonzero b (that corresponds to 0=b which is inconsistent)

Section 3: Vector Equations

- **Column vector** (or just **vector**): a matrix with only one column
- Two vectors are **equal** if their corresponding entries are equal
- For vectors **y,** $\boldsymbol{v_n}$, and scalars c_n – if $\boldsymbol{y} = c_1\boldsymbol{v_1} + \cdots + c_n\boldsymbol{v_n}$ then y is a **linear combination** of vectors v with **weights** c
- Span $\{\boldsymbol{v_1}\ldots\boldsymbol{v_n}\}$ is the collection of all vectors that can be written as $c_1\boldsymbol{v_1} + \cdots + c_n\boldsymbol{v_n}$
- Is **b** in Span $\{\boldsymbol{v_1}\ldots\boldsymbol{v_n}\}$**?** This is the same thing as asking if $x_1\boldsymbol{v_1} + \cdots + x_n\boldsymbol{v_n} = \boldsymbol{b}$ has a solution

Section 4: The Matrix Equation Ax=b

- If A is an m×n matrix with columns $\boldsymbol{a_1 \ldots a_n}$ and **x** is in $\mathbb{R}^n$, then

$$A\mathbf{x} = [\boldsymbol{a_1 \ldots a_n}] \begin{bmatrix} \boldsymbol{x_1} \\ \vdots \\ \boldsymbol{x_n} \end{bmatrix} = \boldsymbol{x_1 a_1} + \cdots + \boldsymbol{x_n a_n}$$

- Ax=b has a solution (*existence*) iff b is a linear combination of the columns of A
- Properties of Ax:
  - A(u + v) = A(u) + A(v)
  - A(cu)=cA(u)

Section 5: Solution Sets of Linear Systems

- A system is **homogeneous** if it can be written as Ax = 0; there is always a zero solution (a **trivial** solution)
- The homogeneous equation Ax = 0 has a nontrivial solution iff the equation has at least one free variable
- Implicit description: the original equations. Explicit description: parametric vector equation
- If Ax = b has a solution, then the solution set is obtained by translating the solution set of Ax = 0, using any particular solution p of Ax = b for the translation

Section 7: Linear Independence

- A set of vectors $\{\boldsymbol{v_1 \ldots v_n}\}$ is **linearly independent** if the vector equation $x_1\boldsymbol{v_1} + \cdots + x_n\boldsymbol{v_n} = \mathbf{0}$ has only the trivial solution; otherwise, the set is **linearly dependent**
- The columns of a matrix A are linearly independent iff the equation Ax = 0 has <u>only</u> the trivial solution
- A set of two vectors is linearly dependent if one of the vectors is a multiple of the other; linearly independent otherwise
- A set of two or more vectors is linearly dependent iff at least one of the vectors in the set is a linear combination of the others
- If a set contains more vectors than there are entries in each vector, then the set is linearly independent
- If a set contains the zero vector, then it is linearly dependent

Section 8: Introduction to Linear Transformations

- A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns for each vector x in $\mathbb{R}^n$ a vector T(x) in $\mathbb{R}^m$
- $\mathbb{R}^n$ is the **domain** of T, $\mathbb{R}^m$ is the **codomain** of T
- Set of all images T(x) is the **range** of T
- Matrix transformation: T(x) is computed as Ax
- A transformation T is **linear** if:
  - T(u + v) = T(u) + T(v) for all u, v in the domain of T
  - T(cu) = cT(u) for all scalars c and all u in the domain of T

- If T is a linear transformation, then T(0) = 0 (maps the zero vector to the zero vector)

Section 9: The Matrix of a Linear Transformation

- For linear transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ there exists a unique m×n matrix A such that T(x)=Ax for all x in $\mathbb{R}^n$; A is called the **standard matrix** for a linear transformation
- A = [$T(\boldsymbol{e_1})$ ... $T(\boldsymbol{e_n})$ **]**
- A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is an image of at least one x in $\mathbb{R}^n$ – there exists at least one solution to T(x)=b; Ax=b is consistent (no zero rows)
- A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is an image of at most one x in $\mathbb{R}^n$ – T(x)=b has a unique solution or no solution; T(x)=0 has only the trivial solution
- For T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$, with A being the standard matrix of the transformation:
  - T maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$ (pivot in every row)
  - T is one-to-one iff the columns of A are linearly independent (pivot in every column)

Chapter 2: Matrix Algebra (sections 1,2,3,8,9)

Section 1: Matrix Operations

- The **diagonal entries** of an m×n matrix A = [a_{ij}] are a_{11}, a_{22}, a_{33}, etc. and they form the **main diagonal** of A
- A **diagonal matrix** is an n×n square matrix whose nondiagonal entries are zero
- An m×n matrix all of whose entries are zero is the **zero matrix**
- Two matrices are equal if they have the same size and their corresponding columns are equal
- Properties of matrix addition:
  - A + B = B + A
  - (A + B) + C = A + (B + C)
  - A + 0 = A
  - r(A + B) = rA + rB
  - (r + s)A = rA + sA
  - r(sA) = (rs)A
- If A is an m×n matrix and B is an n×p matrix, AB = A[b_1 b_2 ... b_p] = [Ab_1 Ab_2 ... Ab_p]
- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B (for entry ij of the matrix AB, multiply row i of A by column j of B)
- AB has same number of rows as A and same number of columns as B
- Properties of matrix multiplication (for A m×n matrix)
  - A(BC) = (AB)C
  - A(B+C) = AB + AC
  - (B+C)A = BA + CA
  - r(AB) = (rA)B = A(rB)
  - $I_m A = A = A I_n$

- Given an m×n matrix A, the **transpose** of A is the n×m matrix A^T whose columns are formed by the corresponding rows of A
- Properties of transpose:
  - $(A^T)^T$ = A
  - $(A+B)^T = A^T + B^T$
  - For any scalar r, $(rA)^T$ = rA^T
  - $(AB)^T = B^T A^T$
- A and B **commute** with each other if AB = BA (not true in general cases)

Section 2: The Inverse of a Matrix

- An n×n matrix A is **invertible** if there exists an n×n matrix C for which AC=I and CA=I
- A **singular matrix** is not invertible; a **nonsingular matrix** is invertible.
- For a 2×2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$; if det(A) = 0, the matrix is not invertible
- If A is an invertible n×n matrix, then for each b in $\mathbb{R}^n$ the equation Ax = b has the unique solution x = A^{-1}b
- Properties of the matrix inverse:
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A^T)^{-1} = (A^{-1})^T$
- The product of two invertible matrices is invertible
- An n×n matrix A is invertible iff it is row equivalent to I_n
- To find the inverse of A, write $[A \mid I_n] \rightarrow row\ reduce\ to\ [\ I_n \mid A^{-1}]$

Section 3: Characteristics of Invertible Matrices

- Invertible matrix theorem: for any given square nxn matrix A, the following are either all true or all false
  - A is invertible
  - A is row equivalent to the nxn identity matrix
  - A has n pivot positions
  - The equation Ax=0 has only the trivial solution
  - The columns of A form a linearly independent set
  - The linear transformation x → Ax is one-to-one
  - The equation Ax = b has at least one solution for each b in $\mathbb{R}^n$
  - The columns of A span $\mathbb{R}^n$
  - The linear transformation x → Ax maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
  - There is a square nxn matrix C such that CA=I
  - There is a square nxn matrix D such that AD = I
  - A^T is an invertible matrix
- A linear transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there exists a function S: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ such that S(T(x)) = x and T(S(x)) = x for all x in $\mathbb{R}^n$ (S is the inverse of T; notation: T^{-1})
- T: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible iff its standard matrix A is invertible; $T^{-1}(x) = A^{-1}x$

Section 8: Subspaces of $\mathbb{R}^n$

- A **subspace** of $\mathbb{R}^n$ in any set H has 3 properties:
  - The zero vector is in H
  - For each u and v in H, u+v is in H (vector addition)
  - For each u in H and each scalar c, the vector cu is in H
- Span $\{\boldsymbol{v_1 \ldots v_n}\}$ is the subspace spanned by $\boldsymbol{v_1 \ldots v_n}$
- The **zero subspace** is the set consisting of only the zero vector
- The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A; it's the space of all vectors b for which Ax = b is solvable
- The column space of an mxn matrix is a subspace of $\mathbb{R}^m$
- The **null space** of a matrix A is the set Nul A of all solutions of the homogeneous equation Ax=0
- The null space of an mxn matrix A is a subspace of $\mathbb{R}^n$
- A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H
- The set $\{\boldsymbol{e_1 \ldots e_n}\}$ is the **standard basis** for $\mathbb{R}^n$
- To find basis for column space: row reduce matrix to find its pivot columns; the set of the pivot columns of the ORIGINAL matrix (before row reduction) is the column space
- To find basis for null space: find solution of Ax = 0; the set of vectors in parametric vector form of the solution is the basis for null space
- (not in chapter): **row space** of A is the set Row A of all linear combinations of the rows of A
- (not in chapter): to find the basis for row space, row reduce the matrix to reduced row echelon form. The nonzero rows will form the basis for Row A
- (not in chapter): **left null space** of A is the null space of A^T
- (not in chapter): to find the basis for left null space, first write [A | I_m] then row reduce so that it becomes [rref(A) | M]; every row in M corresponding to a zero row in rref(A) is a basis vector for LNul A
- (not in chapter): left null space is orthogonal to column space; row space is orthogonal to null space

Section 9: Dimension and Rank

- Suppose that $\beta = \{b_1 \ldots b_p\}$ is a basis for subspace H. For each x in H, the **coordinates** of x relative to H are the weights $c_1 \ldots c_p$ such that x = $c_1 b_1 + \cdots + c_p b_p$ and the vector in $\mathbb{R}^p$ $[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ is the β**-coordinate vector of x**
- The **dimension** of a nonzero subspace H (dim H) is the number of vectors in basis of H. The dimension of the zero subspace is defined as zero.
- The **rank** of a matrix A is the dimension of the column space of A (so, the number of pivot points)
- If a matrix A has n columns, rank A + dim Nul A = n
- If H is a p-dimensional subspace of $\mathbb{R}^n$, any linearly independent set of exactly p elements in H is a basis of H

- Invertible matrix theorem cont. (for a square nxn matrix A these are all true or all false)
    - Columns of A form a basis of $\mathbb{R}^n$
    - Col A = $\mathbb{R}^n$
    - dim Col A = n
    - rank A = n
    - Nul A = {0}
    - dim Nul A = 0

Chapter 3: Determinants (Sections 1,2,3)

Section 1: Introduction to Determinants

- The **determinant** of an nxn matrix A=$[a_{ij}]$ is the sum of n terms of the form $\pm a_{ij}\det A_{ij}$; so det A = $\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$
- Given A=$[a_{ij}]$, the **(i, j)-cofactor** of A $C_{ij} = (-1)^{1+j}\det A_{ij}$
- The determinant of an nxn matrix can be computed by a cofactor expansion across any row or down any column:
    - det A = $a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$
    - det A = $a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$
- cofactor signs: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ etc.
- If A is a diagonal matrix, then det A is the product of the entries on its main diagonal

Section 2: Properties of Determinants

- For square matrix A:
    - If a multiple of one row of A is added to another row to produce B, then det B = det A
    - If two rows of A are interchanged to produce B, then det B = - det A
    - If one row of A is multiplied by k to produce B, then det B = k det A
- For matrix A and its corresponding matrix U in rref; r is the number of times rows have been swapped places:
    - det A = $\begin{cases} \boldsymbol{(-1)^r(product\ of\ pivots\ in\ U)} : & when\ A\ is\ invertible \\ \boldsymbol{0} : & when\ A\ is\ not\ invertible \end{cases}$
- A square matrix A is invertible only if det A ≠ 0
- If A is an nxn matrix, det A^T = det A
- If A and B are nxn matrices, det AB = (det A)(det B)
- $\det A^{-1} = \frac{1}{\det A}$

Section 3: Carmer's Rule, Volume, and Linear Transformations

- **Cramer's rule:** let A be an invertible nxn matrix. For any b in $\mathbb{R}^n$, the unique solution x of Ax=b has entries given by
  - $x_i = \frac{\det A_i(b)}{\det A}$ $i = 1,2 \ldots n$
  - Here, $A_i(b)$ means that the ith column of the matrix A is replaced by the vector b
- The **adjugate** (or **classical adjoint**) of A (adj A) is the matrix of cofactors $\begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$
- If A is an invertible nxn matrix, $A^{-1} = \frac{1}{\det A} adj(A)$
- For a 2x2 matrix A, the area of the parallelogram determined by the columns of A is |detA|
- For a 3x3 matrix A, the volume of the parallelepiped determined by the columns of A is |detA|
- Let T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2x2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then {area of T(S)}= |detA| {area of S}
- Let T: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation determined by a 3x3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then {volume of T(S)}= |detA| {volume of S}

Chapter 4: Vector Spaces (Sections 1,2,3,4,5,6,7)

Section 1: Vector Spaces and Subspaces

- A **vector space** is a nonempty set V of objects (vectors) on which addition and multiplication by scalars is defined. Following axioms hold true for all vector spaces, with u, v, and w in V and scalars c, d
  - u + v is in V
  - u + v = v + u
  - (u + v) + w = u + (v + w)
  - There is a zero vector 0 in V such that u + 0 = 0
  - For each u in V, there is a vector –u in V such that u + (-u) = 0
  - cu is in V
  - c (u + v) = cu + cv
  - (c + d) u = cu + du
  - c(du) = (cd)u
  - 1u = u
- A **subspace** of a vector space Vis a subset H of V that has three properties:
  - The zero vector of V is in H
  - H is closed under vector addition (for each u and v in H, u+v is also in H)
  - H is closed under scalar multiplication (for each u in H, every cu is in H)
- A subspace H of V is itself a vector space
- Every vector space is a subspace (of itself and possibly other larger space)
- The set consisting of only the zero vector in a vector space V is called the **zero subspace**

- If $v_1 \ldots v_p$ are in a vector space V, then Span$\{v_1 \ldots v_p\}$ is a subspace of V
- A **spanning set** for subspace H is $v_1 \ldots v_p$ such that H= Span$\{v_1 \ldots v_p\}$

Section 2: Null Spaces, Column Spaces, and Linear Transformations

- The **null space** of an mxn matrix A is the set of all solutions of Ax=0 and is a subspace of $\mathbb{R}^n$
- The **column space** of an mxn matrix A is the set of all linear combinations of the columns of A; it is a subspace of $\mathbb{R}^m$
- The columns space of an mxn matrix A is all of $\mathbb{R}^m$ iff Ax=b has a solution for every b
- A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector x in V a unique vector T(x) in W such that
    - T(u+v) = T(u) + T(V)
    - T(cu) = c T(u)

Section 3: Linearly Independent Sets; Bases

- An indexed set $\{v_1 \ldots v_p\}$ of two or more vectors (none the zero vector) is linearly dependent iff some v_j is a linear combination of the preceding vectors $v_1 \ldots v_{j-1}$
- Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{b_1 \ldots b_p\}$ in V is a basis for H if
    - β is a linearly independent set
    - The subspace spanned by β coincides with H
- The standard basis for $\mathbb{P}_n$ is $\{1,t,t^2\ldots\}$
- Let S = $\{v_1 \ldots v_p\}$ be a set in V and H = Span $\{v_1 \ldots v_p\}$
    - If one of the vectors in S is a linear combination of other vectors, then the set formed by S after removing that vector still spans H
    - If H≠{0}, some subset of S is a basis for H

Section 4: Coordinate Systems

- Let $\beta = \{b_1 \ldots b_n\}$ be a basis for a vector space V. Each vector x in V can be expressed as x = $c_1 b_1 + \cdots + c_n b_n$
- The β-coordinates of x are the weights $c_1 \ldots c_n$
- $[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the β-coordinate vector of x
- Mapping x → $[x]_\beta$ is the **coordinate mapping** determined by β
- Let $P_\beta = [b_1 \ldots b_n]$, P_β a **change of coordinates matrix.** x = $c_1 b_1 + \cdots + c_n b_n$ is the same thing as saying $x = P_\beta [x]_\beta$
- Let β be a basis for a vector space V. Then the coordinate mapping x → $[x]_\beta$ is a one-to-one linear transformation onto $\mathbb{R}^n$

Section 5: The Dimension of a Vector Space

- If a vector space V has a basis $\beta = \{b_1 \ldots b_n\}$ then any set in V containing more than n vectors must be linearly dependent
- If a vector space V has a basis of n vectors, every basis of V will contain n vectors
- V is **finite-dimensional** if it is spanned by a finite set; V is **infinite-dimensional** if it is spanned by an infinite set
- If H is a subspace of a finite-dimensional vector space V, H is finite-dimensional and any linearly independent set in H can be expanded to a basis for H. Also, dim H ≤ dim V
- Let V be a p-dimensional subspace. Any linearly independent set of exactly p vectors in V is automatically a basis for V

Section 6: Rank

- The **row space** of A is the set of all linear combinations of the row vectors
- If A and B are row equivalent, their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of B and of A

Section 7: Change of Basis

- Let $\beta = \{b_1 \ldots b_n\}$ and C= $\{c_1 \ldots c_n\}$ be bases for a vector space V. Then there is a unique nxn matrix $P_{C \leftarrow \beta}$ such that
  - $[x]_C = P_{C \leftarrow \beta}[x]_\beta$
  - The columns of $P_{C \leftarrow \beta}$ are the C-coordinate vectors of the vectors in basis β: $P_{C \leftarrow \beta} = [\,[b_1]_C\ \ [b_2]_C \ldots [b_n]_C\,]$
- $P_{C \leftarrow \beta}$ is the **change of coordinates matrix** from β to C
- $(P_{C \leftarrow \beta})^{-1} = P_{\beta \leftarrow C}$
- $[c_1\ c_2 \mid b_1\ b_2\,] \rightarrow row\ reduce\ to\ [\,I \mid P_{C \leftarrow \beta}\,]$

Chapter 5: Eigenvalues and Eigenvectors (Sections 1,2,3,4,5)

Section 1: Eigenvectors and Eigenvalues

- An **eigenvector** of an nxn matrix is a nonzero vector x such that Ax = λx for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x for Ax = λx; such that x is called the eigenvector corresponding to λ.
- The null space of the matrix A – λI is called the **eigenspace** of A corresponding to λ
- The eigenvalues of a triangular matrix are the main entries on its diagonal
- If $v_1 \ldots v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1 \ldots \lambda_r$ of an nxn matrix A, then the set $\{v_1 \ldots v_r\}$ is linearly independent

Section 2: The Characteristic Equation

- Invertible Matrix Theorem continued: A is invertible iff
  - The number 0 is not an eigenvalue of A
  - The determinant of A is not zero
- A scalar λ is an eigenvalue of an nxn matrix iff λ satisfies the **characteristic equation** det(A - λI) = 0
- For nxn matrix A, det(A - λI) is an n-degree **characteristic polynomial**
- The **multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation
- If A and B are nxn matrices, A and B are **similar** if there is an invertible matrix P such that $PAP^{-1} = B$ and conversely $P^{-1}BP = A$
- If nxn matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities
- Similarity is not the same thing as row equivalence – performing row operations on a matrix usually changes its eigenvalues

Section 3: Diagonalization

- A is **diagonoalizable** if it is similar to a diagonal matrix D: $A = PDP^{-1}$
- An nxn matrix is diagonalizeable iff it has n linearly independent eigenvectors
- The columns of P are eigenvectors of A; the diagonal entries of D are the eigenvalues corresponding to the order in P
- An nxn matrix A is diagonalizable iff there are enough eigenvectors to form an eigenvector basis of $\mathbb{R}^n$
- An nxn matrix with n distinct eigenvalues is diagonalizable
- For matrices whose eigenvalues are not distinct: an nxn matrix A is diagonalizable iff the sum of the dimensions of the eigenspaces of its eigenvalues equals n (so if the dimensions of the eigenspaces are equal to the multiplicity of their eigenvalues in the characteristic equation)

Section 4: Eigenvectors and Linear Transformations

- $[T(x)]_C = M[x]_\beta$ $where$ $M =$ [$[T(b_1)]_C$ $[T(b_2)]_C$ ... $[T(b_n)]_C$] ; M is the **matrix for T relative to β** or **β-matrix for T**
- Suppose $A = PDP^{-1}$ where D is a diagonal nxn matrix. If β is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the β-matrix for the transformation x → Ax

Section 5: Complex Eigenvalues

- A complex scalar λ satisfies det(A - λI) = 0 iff there is a nonzero vector in $\mathbb{C}^n$ such that Ax = λx; λ is a **complex eigenvalue** and x is a **complex eigenvector**
- The **real** and **imaginary parts** of a complex vector x are the vectors Re x and Im x in $\mathbb{R}^n$ formed from the real and the imaginary parts of the entries of x
- When A is real, its complex eigenvalues occur in conjugate pairs (a ± bi)

- Let A be a real 2x2 matrix with a complex eigenvalue λ = a – bi (b≠0) and an associated eigenvector v in $\mathbb{C}^2$. Then, A= PCP^{-1} where P = [Re(v) Im(v)] and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Chapter 6: Orthogonality and Least Squares (sections 1,2,3,4,5,6,7)

Section 1: Inner Product, Length, and Orthogonality

- The number $u^T v$ is called the **inner product** (or **dot product)** of u and v.
  $$[u_1 \;\; u_2 \;...\;\; u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \;u_1\, v_1 + \;u_2\, v_2 + \cdots + \;u_n\, v_n$$
- Properties of the inner product:
  - $u \cdot v = v \cdot u$
  - $(u + v) \cdot w = u \cdot w + v \cdot w$
  - $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
  - $u \cdot u \geq 0 \; and \; u \cdot u = 0 \; only \; if \; u = 0$
- The **length** (or **norm)** of **v** is the nonnegative scalar **||v||** defined by $||v|| = \sqrt{(v_1)^2 + (v_2)^2 + \cdots + (v_n)^2}$ and $||v||^2 = v \cdot v$
- $||cv||$=$|c|$$||v||$
- A **unit vector** is a vector of length one; to **normalize** a vector, divide it by its norm
- For u and v in $\mathbb{R}^n$ the **distance** between u and v is the length of the vector u – v: $dist(u, v) = ||u - v||$
- Two vectors u and v in $\mathbb{R}^n$ are **orthogonal** to each other if $u \cdot v = 0$
- **Pythagorean theorem**: two vectors are orthogonal iff $||u + v||^2 = ||u||^2 + ||v||^2$
- If a vector v is orthogonal to every vector in a subspace W, then it is **orthogonal to** W
- The set of all vectors orthogonal to a subspace W is called the **orthogonal component** of W and is denoted by $W^\perp$
- A vector x is in $W^\perp$ if it is orthogonal to every vector in a set that spans W
- $W^\perp$ is a subspace of $\mathbb{R}^n$
- Let A be an mxn matrix. Orthogonal component of the row space of A is the null space of A and the orthogonal component of the column space of A is the null space of A^T
- In $\mathbb{R}^2$ and $\mathbb{R}^3$: $u \cdot v = ||u||\,||v|| \cos\theta$

Section 2: Orthogonal Sets

- A set of vectors $\{u_1 \; ... \; u_p\}$ is an **orthogonal set** in $\mathbb{R}^n$ if each pair of distinct vectors from the set is orthogonal: $u_i \cdot u_j = 0$ whenever i≠j
- If S = $\{u_1 \; ... \; u_p\}$ is a set of orthogonal nonero vectors in $\mathbb{R}^n$ then S is linearly independent and hence is a basis for the subspace spanned by S
- An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set

- Let $\{u_1 \ldots u_p\}$ be an orthogonal basis for a subspace W of of $\mathbb{R}^n$. For each y in W, the weights of the linear combination $y = c_1u_1 + \cdots + c_pu_p$ are given by $c_j = \frac{y \cdot u_j}{u_j \cdot u_j}$
- A vector y in $\mathbb{R}^n$ can be written as the sum of a multiple of a vector u and an orthogonal component to u $y = \hat{y} + z$. The vector $\hat{y}$ is called the **orthogonal projection of y onto u** and the vector z is called the **component of y orthogonal to u**
- A projection is determined by the subspace L spanned by u; $\hat{y} = proj_L y = \frac{y \cdot u}{u \cdot u} u$
- A set of vectors $\{u_1 \ldots u_p\}$ is an **orthonormal** set if it is an orthogonal set of unit vectors. If W is spanned by such a set, then $\{u_1 \ldots u_p\}$ is an **orthonormal basis** for W
- An mxn matrix U has orthonormal columns iff $U^T U = I$
- Let U be an mxn matrix with orthonormal columns and let y and x be in $\mathbb{R}^n$. Then,
    - $||Ux|| = ||x||$
    - $(Ux) \cdot (Uy) = x \cdot y$
    - $(Ux) \cdot (Uy) = 0$ iff $x \cdot y = 0$
- An **orthogonal matrix** is a square invertible matrix U such that $U^{-1} = U^T$. Such a matrix has orthonormal columns and orthonormal rows.

Section 3: Orthogonal Projections

- **The Orthogonal Decomposition Theorem:**
    - Let W be a subspace of $\mathbb{R}^n$. Then each y in $\mathbb{R}^n$ can be written uniquely in the form $y = \hat{y} + z$ where $\hat{y}$ is in W and z is orthogonal to W. In fact, if $\{u_1 \ldots u_p\}$ is any orthogonal basis of W, then $\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$ and z = y-$\hat{y}$
- $\hat{y}$ is the **orthogonal projection of y onto W**
- If $\{u_1 \ldots u_p\}$ is an orthonormal basis, then $proj_W y = (y \cdot u_1)u_1 + \cdots + (y \cdot u_p)u_p$; if U = [$u_1 \ldots u_p$] then $proj_W y = UU^T y$ for all y in $\mathbb{R}^n$

Section 4: The Gram-Schmidt Process

- Gram-Schmidt is used to derive an orthogonal basis for a subspace from a given non-orthogonal basis
- Given a basis $\{x_1\ x_2 \ldots x_p\}$ for a nonzero subspace W of $\mathbb{R}^n$
    - $v_1 = x_1$
    - $v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$
    - $v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$
    - ...
    - $v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$
    - Then $\{v_1\ v_2 \ldots v_p\}$ is an orthogonal basis for W

Section 5: Least-Squares Problems

- If A is mxn and b is in $\mathbb{R}^m$, a **least-squares solution** of Ax=b is an $\hat{x}$ in $\mathbb{R}^n$ such that $\left||b - A\hat{x}|\right| \leq \left||b - Ax|\right|$ for all x in $\mathbb{R}^n$
- The set of least-squares solutions of Ax=b coincides with the nonempty set of solutions of the normal equation $A^T Ax = A^T b$
- $\hat{x} = (A^T A)^{-1} Ab$
- The distance from b to A$\hat{x}$ is called the **least-squares error** of this approximation

Section 6: Applications to Linear Models

- Least-squares lines: $y = \beta_0 + \beta_1 x$ (x and y from experimental data)

Chapter 7: Symmetric Matrices and Quadratic Forms

Section 1: Diagonalization of Symmetric Matrices

- A **symmetric** matrix is a matrix such that $A = A^T$
- If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal
- An nxn matrix A is **orthogonally diagonalizable** ($A = PDP^T = PDP^{-1}$) iff A is a symmetric matrix
- **Spectral theorem for symmetric matrices:** an nxn symmetrix matrix A has the following properties:
    - A has n real eigenvalues, counting multiplicities
    - The dimension of the eigenspace of each eigenvalue equals its multiplicity as a root of the characteristic equation
    - The eigenspaces are mutually orthogonal (eigenvectors corresponding to different eigenvalues are orthogonal
    - A is orthogonally diagonalizable

True/False Review

Chapter 1

(k) If A is an m×n matrix and the equation Ax=b is consistent for every b in $\mathbb{R}^m$, then A has m pivot columns:
TRUE – for Ax=b to be consistent for every b there needs to be a pivot in every row

(o) If A is an m×n matrix, of the equation Ax=b has at least two different solutions, and if the equation Ax=c is consistent, then Ax=c has many solutions:
TRUE – the two are translations of one another, have the same number of solutions (so, many solutions)

(u) If u, v, and w are nonzero vectors in $\mathbb{R}^2$, then w is a linear combination of u and v:
FALSE – if u and v are multiples of one another, w will not be a linear combination of u & v

(w) Suppose that $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$ are in $\mathbb{R}^5$, $\boldsymbol{v_2}$ not a multiple of $\boldsymbol{v_1}$**,** and $\boldsymbol{v_3}$ is not a linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$**.** Then, $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is linearly independent
FALSE – if one of the vectors is the zero vector, the set would be linearly dependent by definition

(z) If A is an m×n matrix with m pivot columns, then the linear transformation x→Ax is one-to-one:
FALSE – for the transformation to be one-to-one, the standard matrix needs to have a pivot in every column. In this case, it has a pivot in every row, which means that it would be one-to-one only if m=n

Chapter 2

(b) If AB=C and C has 2 columns, then A has 2 columns.
False: C will have the same number of rows as A and the same number of columns as B

(c) Left-multiplying a matrix B by a diagonal matrix A with nonzero entries on the diagonal, scales the rows of B
True: each row of A will only have one nonzero entry in each row and that entry will scale the rows of B

(e) If AC = 0, then either A = 0 or C = 0
False: a row vector and a column vector can have nonzero entries and still give zero as a result of multiplication (for instance, row vector [1 1] and column vector [1 -1])

(f) If A and B are nxn, then (A + B) (A – B) = $A^2 - B^2$
False: $(A+B)(A-B) = A^2 - B^2 - AB + BA$; since matrix multiplication is NOT commutative, AB ≠ BA

(l) If AB = I, then A is invertible

False: this statement does not specify whether or not A is a square matrix. There could be a case where A is an nxm matrix and B is an mxn matrix, and their product results in the identity matrix. Only square matrices can be invertible.

(m) If A and B are square and invertible, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$

False: this statement does not state that A and B are the same size (could be nxn and mxm in which case their multiplication would not make any sense). Even if we assume that A and be have the same size, $(AB)^{-1} = B^{-1}A^{-1}$, in reverse order of what was stated

(n) If AB = BA and if A is invertible, then $A^{-1}B = BA^{-1}$

True: take AB = BA and multiply both sides on the left by A^{-1}, getting B = A^{-1} BA. Then multiply on the right by A^{-1} which gives you BA^{-1} = A^{-1}B

(p) If A is a 3 x 3 matrix and the equation Ax = $\begin{bmatrix}1\\0\\0\end{bmatrix}$ has a unique solution, then A is invertible.

True: for the system to have a unique solution, the homogeneous case Ax=0 must only have one solution, which means that there is a pivot in every row of the matrix A. As a result, the matrix is invertible.

Chapter 3

(a) If A is a 2x2 matrix with a zero determinant, then one column of A is a multiple of the other

True: a zero determinant implies that the matrix is not invertible, which means that its columns are linearly dependent.

(c) If A is a 3x3 matrix, then det 5A = 5 det A

False: det 5A means that every row of A is multiplied by 5. Recall the rule that if one row of A is multiplied by k to get matrix B, then k detA=detB. This means that det 5A = 125 det A

(g) If B is produced by multiply row 3 of A by 5, then det B = 5 det A

True: if one row of A is multiplied by k to get matrix B, then det B = k det A

(i) $\det A^T = -\det A$

False: $\det A^T = \det A$

(k) $\det A^T A \geq 0$

True: $\det A^T A = \det A^T \det A$; now recall that $\det A^T = \det A$, which means that $\det A^T A = (\det A)^2$. If A is not invertible, detA = 0. If A is invertible, $(\det A)^2 > 0$

(l) Any system of n linear equations in n variables can be solved by Cramer's rule

False: Cramer's rule can only be applied if the nxn matrix of the system is invertible; in other words, the system of linear equations must form a linearly independent set

(n) If $A^3 = 0$, then det A = 0
True: $\det A^3 = \det 0 = 0$; since $\det AB = \det A \det B$, we can say that $\det A^3 = (\det A)^3$ and since $\det A^3 = 0$, det A = 0

(p) If A is invertible, then $(\det A)(\det A^{-1}) = 1$
True: $\det A^{-1} = \frac{1}{\det A}$

Chapter 4

(a) The set of all linear combinations of $v_1 \ldots v_p$ is a vector space
True: such set constitutes Span $\{v_1 \ldots v_p\}$

(c) For S=$\{v_1 \ldots v_p\}$, if $\{v_1 \ldots v_{p-1}\}$ is linearly independent, then so is S
False: v_p could be a linear combination of its preceding vectors in which case S is not linearly independent

(f) For vector space V and subspace S=$\{v_1 \ldots v_p\}$, if dim V = p and Span S = V, then S cannot be linearly dependent
True: since S (which has p components) spans the p-dimensional vector space V, all the vectors in the set S must be linearly independent because in order to span V, S has to have p linearly independent components

(h) The nonpivot columns of a matrix are always linearly dependent
False: there could be nonpivot columns that are linearly independent

(j) Row operations on a matrix can change its null space
False: row operations on A do not change the solutions to Ax=0

(l) If an mxn matrix A is row equivalent to an echelon matrix U, and if U has k nonzero rows, then the dimension of the solution space of Ax=0 is m-k
False: if U has k nonzero rows, rank A = k. We know that rank A + dim Nul A = n, NOT m; therefore, the dimension of the null space of A equals n-k

(q) If A is mxn and rank A = m, then the linear transformation x→Ax is one-to-one
False: to be one to one, rank A would need to be n (number of columns), not m (number of rows) – in matrix form, a pivot in every column

(r) If A is mxn and the linear transformation x→Ax is onto, then rank A = m
True: for the transformation to be onto, rank A must be the number of rows, m (in matrix form, a pivot in every row)

(s) A change-of-coordinates matrix is always invertible
True: since the columns of the change-of-coordinates matrix are basis vectors, they are by definition linearly independent which means that the matrix is square has a pivot in every row and column, meaning that the matrix is invertible.

Chapter 5

(a) If A is invertible and 1 is an eigenvalue of A, then 1 is also an eigenvalue of A^{-1}
True: if Ax=1x and we left-multiply both sides by A^{-1}, we get A^{-1}x = 1x which means that 1 is an eigenvalue of A^{-1}

(b) If A is row equivalent to the identity matrix I, then A is diagonalizable
False: being row equivalent to the identity matrix makes a matrix invertible; not all invertible matrices are diagonalizable.

(c) If A contains a row or column of zeroes, then 0 is an eigenvalue for A
True: if A contains a row or column of zeroes, it is not invertible and all noninvertible matrices have zero as an eigenvalue

(e) Each eigenvector of A is also an eigenvector of A^2
True: If given Ax = λx, left multiplying both sides by A we get A^2x = λAx which then follows as A^2x = λ^2x. This means that x is an eigenvector for both A and A^2

(i) Two eigenvectors corresponding to the same eigenvalue are always linearly dependent
False: an eigenvalue with a multiplicity greater than zero could have several linearly independent eigenvectors

(l) The sum of two eigenvectors of a matrix A is also an eigenvector of A
False: the sum of two eigenvectors generally is not an eigenvector

(n) The matrices A and A^T have the same eigenvalues, counting multiplicities
True: matrices A and A^T have the same characteristic equation

(q) If A is diagonalizable, then the columns of A are linearly independent
False: if columns of A are linearly independent, the matrix is invertible; a matrix does not have to be invertible to be diagonalizable

(x) If A is an nxn diagonalizable matrix, then each vector in $\mathbb{R}^n$ can be written as a combination of eigenvectors of A
True: since A is diagonalizable, its eigenvectors form an eigenbasis for $\mathbb{R}^n$

Chapter 6

(f) If x is orthogonal to both u and v, then x must be orthogonal to u-v
True: if xu=0 and xv=0, then xu-xv=0 and x(u-v)=0 meaning that x is orthogonal to u-v

(h) If $||u-v||^2 = ||u||^2 + ||v||^2$ then u and v are orthogonal
True: the Pythagorean Theorem states that u and v are orthogonal if $||u+v||^2 = ||u||^2 + ||v||^2$; in the case given, v is replaced with (-v) and $||-v||^2 = ||v||^2$

(j) If a vector y coincides with its orthogonal projection onto a subspace W then y is in W
True: the orthogonal projection of y onto W is always in W so y is in W

(k) The set of all vectors in $\mathbb{R}^n$ orthogonal to one fixed vector is a subspace of $\mathbb{R}^n$
True

(n) If a matrix U has orthonormal columns, then $UU^T = I$
False: this would be true if the matrix was square

(o) A square matrix with orthogonal columns is an orthogonal matrix
False: the columns of an orthogonal matrix are orthonormal

(p) If a square matrix has orthonormal columns, then it also has orthonormal rows
True: orthogonal matrices have orthonormal columns and rows

(q) If W is a subspace, then $||proj_W v||^2 + ||v - proj_W v||^2 = ||v||^2$
True: $v - proj_W v$ and $proj_W v$ are orthogonal so the given statement is the Pythagorean Theorem

Chapter 7

(a) If A is orthogonally diagonalizable, then it is symmetric
True: only symmetric matrices are orthogonally diagonalizable

(c) If A is an orthogonal matrix, then ||Ax||=||x|| for all x in $\mathbb{R}^n$
True: an orthogonal matrix has orthogonal unit vectors as columns so ||Ax||=||x||

(e) If A is an nxn matrix with orthogonal columns, then $A^T = A^{-1}$
False: for that to happen, the matrix needs to have orthonormal columns